A NEW HYBRID MIXED FINITE ELEMENT METHOD TO SOLVE ACOUSTIC FLUID-STRUCTURE INTERACTION PROBLEMS

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Abstract. *Fluid-structure interactions (FSI) modeling is an important problem with applications in geophysics and engineering, including vibration and structural dynamic response, dam failures during seismic excitation, reductions in noise emissions, noise prediction in aeroacoustic and so on. In this work we develop a new stabilized hybrid mixed finite element method to solve acoustics FSI problems. Hybrid methods are characterized by the introduction of Lagrange multipliers to weakly impose continuity on the interelement interfaces. This approach generates a global system involving only the degrees-of-freedom associated with the multipliers. The quantities of interest are obtained from local problems that are solved at the element level. In this context, to generate the hybrid method for the coupled fluid-structure problem we combine the hybrid formulations for the Helmholtz problem with the time-harmonic elastic wave problem. This methodology allows for natural coupling of fluid-structure interface conditions via Lagrange multipliers. Some numerical experiments results illustrate the flexibility and the robustness of the proposed finite element formulation.*

Keywords: *Fluid-Structure Interactions, Helmholtz Problem, Linear Elasticity Problem, Interface Conditions, Mixed-Hybrid Methods*

1 INTRODUCTION

The dynamic interaction between a fluid and a structure is a significant concern in many engineering problems. These problems include the modeling and simulation of aircraft, rockets, turbines, marine structures (fixed, floating and submerged), storage tanks, dams, and suspension bridges. The interaction may change the dynamic characteristics of the structure and consequently its response to transient and/or periodic excitation (Ross et al. (2008, 2009)).

The acoustic fluid-structure problem is modeled by a coupled system of partial differential equations. Structural acoustics problems in general aim to solve for the acoustic pressure field, resulting in a fluid and solid system due to mechanical solid excitation or external fluid excitation. Fluid behavior is modeled by the Helmholtz equation, while structure behavior is modeled by the time-harmonic elastic wave equation. Several finite element formulations based on displacement, pressure, and potential have been applied to problems involving the interaction between acoustic fluids and elastic structures (Gladwell, 1966; Gladwell and Mason, 1971; Craggs, 1972; Zienkiewicz and Bettess, 1978; Nefske et al., 1982; Luke and Martin, 1995; Ross et al., 2008, 2009; González et al., 2012).

In this work, we propose a new Stabilized dual Hybrid mixed finite element method for the Helmholtz problem (*SHHel*). This method is characterized by weakly imposing the continuity on each interelement edge via Lagrange multipliers and by adding least square residuals similar to the method developed by Igreja et al. (2014) for the Darcy problem. Furthermore, the proposed methodology is able to recover, in a convenient way, the stability of incompatible finite element approximations, such as Lagrangian polynomial approximations of the same order for all fields, which are unstable for the usual dual mixed formulation, as illustrated in Correa and Loula (2008). To solve the time-harmonic elastic wave problem, we propose a new stabilized primal hybrid method, denoted by *SHEW* (Stabilized primal Hybrid formulation for the time-harmonic Elastic Wave problem) with the Lagrange multipliers associated to the displacement field. From the proposed methods for the fluid domain and the structure domain we present a new Stabilized Hybrid method for acoustic Fluid-Structure interaction (*SHFS*) based in previously work developed by Igreja (2015); Igreja et al. (2015). The method SHFS couple the SHHel and SHEW methods and the interface fluid/structure conditions are naturally imposed through the Lagrange multipliers.

The paper is organized as follows. We describe the model problem and introduce some notations and definitions in Section 2. In Section 3 we present the proposed hybrid methods for solving the Helmholtz and the Elastic Wave problems, independently, and the coupled acoustic fluid-structure interaction problem. We make some remarks on the solving methodology in Section 4. In Section 5 some numerical experiments are presented, showing the convergence rates. The paper ends with some concluding remarks in Section 6.

2 PRELIMINARIES

In this section we present the model problem and some definitions and notations commonly adopted to construct variational formulations in broken function spaces associated with hybrid methods.

2.1 FLUID-STRUCTURE MODEL PROBLEM

The domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) for the coupled fluid-structure model problem (see Fig. 1) is composed by a subdomain Ω_f, with outward unit normal $\mathbf{n}_f$, which we identify as the fluid domain, and a subdomain Ω_s, with outward unit normal $\mathbf{n}_s$, that represents the structure domain. The fluid behavior in Ω_f is modeled by the Helmholtz equation, while the solid behavior in Ω_s is described by the time-harmonic elastic wave system. These subdomains are separated by a smooth interface $\Gamma_{fs} = \partial\Omega_f \cap \partial\Omega_s$. The Lipschitz boundaries of the fluid and solid domains are denoted by $\Gamma_f = \partial\Omega_f \setminus \Gamma_{fs}$ and $\Gamma_s = \partial\Omega_s \setminus \Gamma_{fs}$. We denote by $\mathbf{u}_f = \mathbf{u}|_{\Omega_f}$ and $p_f = p|_{\Omega_f}$ the velocity and pressure fields, respectively, in the fluid domain and by $\mathbf{u}_s = \mathbf{u}|_{\Omega_s}$ the displacement vector field of the structure.

We proceed to the presentation of the equations describing the phenomena in each medium.

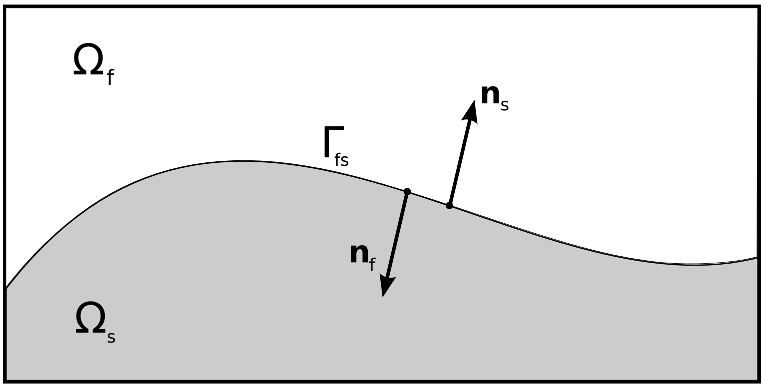


Figure 1: A sketch of the domain for the fluid-structure problem showing the interface of discontinuity.

Helmholtz System

For the fluid domain Ω_f we consider that the propagation of acoustic waves occurs in an ideal compressible fluid. A linear model for this phenomenon is given by the wave equation

$$-\operatorname{div}(\nabla\varphi) + \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = F, \tag{1}$$

where $\varphi(\mathbf{x}, t)$ represents small oscillations of the pressure, c is the velocity of the sound in the acoustic medium and $F(\mathbf{x}, t) = f(\mathbf{x})e^{i\omega_f t}$ is a source term. Considering harmonic solutions in time with circular frequency ω_f, the pressure field is written as $\varphi(\mathbf{x}, t) = p_f(\mathbf{x})e^{i\omega_f t}$ and the pressure amplitude p_f satisfies the Helmholtz equation

$$-\operatorname{div}\left(\nabla p_f\right) - k_f^2 p_f = f, \tag{2}$$

where the parameter $k_f = \omega_f/c$, known as the *wavenumber*, characterizes the oscillatory behavior of the solution φ. This problem can be formulated in two fields, velocity and pressure, by introducing the vector field $\mathbf{u}_f = -\nabla p_f$ and rewriting the Helmholtz equation (2) in a mixed form, as follows.

Given the wavenumber k_f and the function f, find the velocity field $\mathbf{u}_f : \Omega_f \to \mathbb{R}^d$ and the pressure field $p_f : \Omega_f \to \mathbb{R}$ such that

$$\mathbf{u}_f + \nabla p_f = 0 \quad \text{in} \quad \Omega_f, \tag{3}$$

$$\operatorname{div}\mathbf{u}_f - k_f^2 p_f = f \quad \text{in} \quad \Omega_f. \tag{4}$$

This system can be supplemented by a Dirichlet boundary condition

$$p_f = g \qquad \text{on} \quad \Gamma_f \tag{5}$$

or a Robin boundary condition

$$-\mathbf{u}_f \cdot \mathbf{n}_f + ik_f p_f \;=\; r \qquad \text{on} \quad \Gamma_f, \tag{6}$$

where $i = \sqrt{-1}$.

Time-Harmonic Elastic Wave System

The solid domain Ω_s is occupied by an isotropic and linearly elastic body characterized by the real valued constant mass density $\rho_s > 0$ and the Lamé coefficients $\lambda, \mu \in \mathbb{R}$ satisfying $\mu > 0$ and $3\lambda + 2\mu > 0$. In this context, we define the time-harmonic elastic wave problem supplemented by Robin boundary conditions as

Given the mass density ρ_s, the circular frequency ω_s, the tensor $\mathbf{A}$ and the source terms $\mathbf{f}$ *and* $\mathbf{g}$*, find the displacement field $\mathbf{u}_s : \Omega_s \to \mathbb{R}^d$ satisfying*

$$-\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}_s) - \rho_s \omega_s^2 \mathbf{u}_s \;=\; \mathbf{f} \qquad \text{in} \quad \Omega_s \tag{7}$$

$$\boldsymbol{\sigma}(\mathbf{u}_s)\mathbf{n}_s + i\mathbf{A}\mathbf{u}_s \;=\; \mathbf{g} \qquad \text{on} \quad \Gamma_s \tag{8}$$

$\boldsymbol{\sigma}(\mathbf{u}_s)$ is the symmetric Cauchy stress tensor. For a linear, homogeneous and isotropic material $\boldsymbol{\sigma}(\mathbf{u}_s)$ is given by

$$\boldsymbol{\sigma}(\mathbf{u}_s) = \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_s) = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}_s) + \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}_s))\mathbf{I} \tag{9}$$

where $\mathbb{D} = 2\mu\mathbb{I} + \lambda\,\mathbf{I} \otimes \mathbf{I}$ is the isotropic elasticity tensor, $\boldsymbol{\varepsilon}(\mathbf{u}_s) = 1/2(\nabla\mathbf{u}_s + \nabla\mathbf{u}_s^T)$ is the linear strain tensor, $\mathbf{I}$ is the second-order identity tensor, $\mathbb{I}$ is the fourth-order identity tensor on symmetric second-order tensors and $\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}_s) = \operatorname{div}\mathbf{u}_s$. For linear plane strain the Lamé coefficients are given by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}, \tag{10}$$

where E denotes the elasticity modulus and ν is the Poisson's ratio. The tensor $\mathbf{A}$ is defined as

$$\mathbf{A} = \begin{bmatrix} k_p & 0 \\ 0 & k_s \end{bmatrix}, \tag{11}$$

where k_p is the longitudinal (pressure) wavenumber and k_s is the transverse (shear) wavenumber, are shown below

$$k_p = \omega_s \sqrt{\frac{\rho_s}{2\mu + \lambda}}, \tag{12}$$

$$k_s = \omega_s \sqrt{\frac{\rho_s}{\mu}}. \tag{13}$$

Interface Fluid-Structure Conditions

Now we present the interface conditions between the acoustic domain and the structural domain, $\Gamma_{fs} = \partial\Omega_f \cap \partial\Omega_s$. For the acoustic domain, the local balance of linear momentum equation should be satisfied as follows (Yoon et al., 2007; Vicente et al., 2015)

$$\mathbf{u}_f \cdot \mathbf{n}_f + \rho_f \omega_s^2 \mathbf{u}_s \cdot \mathbf{n}_s = 0 \qquad \text{on} \quad \Gamma_{fs}. \tag{14}$$

This equation represents the kinematic compatibility of the normal displacements at the interface of fluid and structural domains. We also have to make sure that the traction on the solid part equals the fluid pressure on the interface:

$$\boldsymbol{\sigma}(\mathbf{u}_s)\mathbf{n}_s + p_f \mathbf{n}_f = 0 \qquad \text{on} \quad \Gamma_{fs}. \tag{15}$$

Equation (15) indicates the action of pressure forces exerted by the fluid on the structure and represents the equilibrium condition at the interface between the domains.

2.2 NOTATIONS AND DEFINITIONS

To introduce the stabilized hybrid formulations we first recall some notations and definitions. Let $H^m(\Omega)$ the usual Sobolev space equipped with the usual norm $\|\cdot\|_{m,\Omega} = \|\cdot\|_m$ and seminorm $|\cdot|_{m,\Omega} = |\cdot|_m$, with $m \geq 0$. For $m = 0$, we induction $L^2(\Omega) = H^0(\Omega)$ as the space of square integrable functions and $H_0^1(\Omega)$ the subspace of functions in $H^1(\Omega)$ with zero trace on $\partial\Omega$.

For a given function space $V(\Omega)$, let $[V(\Omega)]^d$ and $[V(\Omega)]^{d\times d}$ be the spaces of all vector and tensor fields whose components belong to $V(\Omega)$, respectively. Without further specification, these spaces are furnished with the usual product norms (which, for simplicity, are denoted similarly as the norm in $V(\Omega)$). For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ and matrices $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{R}^{d\times d}$ we use the standard notation.

Restricting to the two-dimensional case ($d = 2$), we define a regular finite element partition $\mathcal{T}_h$ of the domain Ω:

$$\mathcal{T}_h = \{K\} := \text{the union of all elements } K.$$

In cases where Ω is divided into subdomains Ω_i with smooth boundary $\partial\Omega_i$ and $\Gamma_i = \partial\Omega \cap \partial\Omega_i$, we have for each subdomain the following regular partition

$$\mathcal{T}_h^i = \{K \in \mathcal{T}_h \cap \Omega_i\},$$

and the following set of edges

$$\mathcal{E}_h^i = \{e;\, e \text{ is an edge of } K, \text{ for at least one } K \in \mathcal{T}_h^i\},$$

$$\mathcal{E}_h^{\partial,i} = \{e \in \mathcal{E}_h^i;\, e \subset \Gamma_i\}$$

$$\mathcal{E}_h^{0,i} = \{e \in \mathcal{E}_h^i;\, e \text{ is an interior edge of } \Omega_i\},$$

$$\mathcal{E}_h^{ij} = \mathcal{E}_h^{0,i} \cap \mathcal{E}_h^{0,j}.$$

This last case denotes the edges that compose the interface between the subdomains, where Ω_i and Ω_j are two adjacent subdomains.

We assume that the domain Ω is polygonal. Thus, there exists $c > 0$ such that $h \leq ch_e$, where h_e is the diameter of the edge $e \in \partial K$ and h, the mesh parameter, is the maximum element diameter. For each element K we associate a unit normal vector $\mathbf{n}_K$. Let $\mathbf{V}_h^l$ and Q_h^m denote broken function spaces on $\mathcal{T}_h$ given by

$$\mathbf{V}_h^l(\Omega) = \{\mathbf{v} \in [L^2(\Omega)]^2;\, \mathbf{v}_h|_K \in [\mathbb{S}_l(K)]^2,\ \forall K \in \mathcal{T}_h\}, \tag{16}$$

$$Q_h^m(\Omega) = \{q \in L^2(\Omega); q_h|_K \in \mathbb{S}_m(K),\ \forall K \in \mathcal{T}_h\}, \tag{17}$$

where $\mathbb{S}_l(K)$ and $\mathbb{S}_m(K)$ denote the space of polynomial functions of degree at most l and m, respectively, on each variable. To introduce the hybrid methods we define the following space associated with the Lagrange multiplier

$$\mathbf{M}_h^n(\mathcal{E}_h) = \{\boldsymbol{\mu} \in [C^0(\mathcal{E}_h)]^2 : \boldsymbol{\mu}|_e = [p_n(e)]^2,\ \forall e \in \mathcal{E}_h^0\}. \tag{18}$$

Similarly, $p_n(e)$ is the space of polynomial functions of degree at most n on an edge e.

3 HYBRID METHODS

In this section we introduce, separately for each subdomain, the hybrid formulation for the Helmholtz and Elastic Wave problems. The first is the method, called SHHel, that is formulated in the fluid domain Ω_f and is characterized by weakly imposing the continuity via Lagrange multipliers related to the velocity field and stabilizing with the addition of least square residuals. For the solid domain Ω_s, we develop the stabilized hybrid SHEW method, which associates a Lagrange multiplier to the displacement field. These methods are coupled using the Lagrange multipliers to naturally impose the interface fluid/structure conditions, giving rise to the SHFS (Stabilized Hybrid formulation for acoustic Fluid-Structure interaction) method.

3.1 Stabilized Hybrid Formulation for the Helmholtz Problem

To introduce the hybrid formulation for the Helmholtz Problem in the fluid domain Ω_f we first consider Eqs. (3)-(4) multiplied by their respective weighting functions and integrated by parts on each element $K \in \mathcal{T}_h^f$, getting the following local weak formulation for $[\mathbf{u}_h^f, p_h^f] \in \mathbf{V}_h^l(\Omega_f) \times Q_h^m(\Omega_f)$

$$\begin{aligned}
\int_K \mathbf{u}_h^f \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} - \int_K p_h^f \operatorname{div} \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \int_{\partial K} p_h^f (\mathbf{v}_h \cdot \mathbf{n}_K) \,\mathrm{d}s &= 0, && \forall \mathbf{v}_h \in \mathbf{V}_h^l(\Omega_f),\\
- \int_K \operatorname{div} \mathbf{u}_h^f\, q_h \,\mathrm{d}\mathbf{x} + \int_K k_f^2\, p_h^f\, q_h \,\mathrm{d}\mathbf{x} &= - \int_K f\, q_h \,\mathrm{d}\mathbf{x}, && \forall q_h \in Q_h^m(\Omega_f).
\end{aligned}$$

To derivate a hybrid method we introduce a Lagrange multiplier $\boldsymbol{\lambda}^f$ defined as the trace of $\mathbf{u}^f$, $\boldsymbol{\lambda}^f = \mathbf{u}^f|_e$, on each edge $e \in \mathcal{E}_h^f$. We also need to add a symmetrization term and a stabilization term for the multiplier on $\partial K \in \mathcal{T}_h^f$, obtaining the following problem:

Given the wavenumber k_f, find $[\mathbf{u}_h^f, p_h^f] \in \mathbf{V}_h^l(\Omega_f) \times Q_h^m(\Omega_f)$ and the Lagrange multiplier $\boldsymbol{\lambda}_h^f \in \mathbf{M}_h^n(\mathcal{E}_h^f)$ such that, for all $[\mathbf{v}_h, q_h] \in \mathbf{V}_h^l(\Omega_f) \times Q_h^m(\Omega_f)$ and $\boldsymbol{\mu}_h \in \mathbf{M}_h^n(\mathcal{E}_h^f)$

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h^f} \int_K \mathbf{u}_h^f \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} - \sum_{K\in\mathcal{T}_h^f} \int_K p_h^f \operatorname{div} \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \sum_{K\in\mathcal{T}_h^f} \int_{\partial K} p_h^f (\mathbf{v}_h \cdot \mathbf{n}_K) \,\mathrm{d}s\\
&+ \sum_{K\in\mathcal{T}_h^f} \int_{\partial K} q_h (\mathbf{u}_h^f - \boldsymbol{\lambda}_h^f) \cdot \mathbf{n}_K \,\mathrm{d}s + \beta_f \sum_{K\in\mathcal{T}_h^f} \int_{\partial K} (\mathbf{u}_h^f - \boldsymbol{\lambda}_h^f) \cdot \mathbf{v}_h \,\mathrm{d}s\\
&- \sum_{K\in\mathcal{T}_h^f} \int_K \operatorname{div} \mathbf{u}_h^f\, q_h \,\mathrm{d}\mathbf{x} + \sum_{K\in\mathcal{T}_h^f} \int_K k_f^2\, p_h^f\, q_h \,\mathrm{d}\mathbf{x} = - \sum_{K\in\mathcal{T}_h^f} \int_K f\, q_h \,\mathrm{d}\mathbf{x},
\end{aligned} \tag{19}$$

$$- \sum_{K\in\mathcal{T}_h^f} \int_{\partial K} p_h^f (\boldsymbol{\mu}_h \cdot \mathbf{n}_K) \,\mathrm{d}s - \beta_f \sum_{K\in\mathcal{T}_h^f} \int_{\partial K} (\mathbf{u}_h^f - \boldsymbol{\lambda}_h^f) \cdot \boldsymbol{\mu}_h \,\mathrm{d}s = 0, \tag{20}$$

where the stabilization parameter β_f is given by

$$\beta_f = \frac{k_f}{h}. \tag{21}$$

Note that the first term in Eq. (20) imposes the continuity of the pressure between the elements and the second term stabilizes the velocity and the Lagrange multiplier.

Also, we add to the system (19)-(20) the least squares stabilization terms related to Eqs. (3) and (4) and to the rotational of Eq. (3) in each element $K \in \mathcal{T}_h^f$ in order to stabilize the local variables $\mathbf{u}_h^f$ and p_h^f (Harari and Hughes, 1992; Monk and Wang, 1999; Loula, 2011). Thus, we derive the stabilized hybrid (SHHel) method supplemented by Robin boundary conditions (6), which can be presented as

Find the pair $[\mathbf{u}_h^f, p_h^f] \in \mathbf{V}_h^l(\Omega_f) \times Q_h^m(\Omega_f)$ *and the Lagrange multiplier* $\boldsymbol{\lambda}_h^f \in \mathbf{M}_h^n(\mathcal{E}_h^f)$ *such that, for all* $[\mathbf{v}_h, q_h] \in \mathbf{V}_h^l(\Omega_f) \times Q_h^m(\Omega_f)$ *and* $\boldsymbol{\mu}_h \in \mathbf{M}_h^n(\mathcal{E}_h^f)$

$$A_{SHHel}([\mathbf{u}_h^f, p_h^f, \boldsymbol{\lambda}_h^f]; [\mathbf{v}_h, q_h, \boldsymbol{\mu}_h]) = F_{SHHel}([\mathbf{v}_h, q_h, \boldsymbol{\mu}_h]), \tag{22}$$

with

$$\begin{aligned} A_{SHHel}([\mathbf{u}_h^f, p_h^f, \boldsymbol{\lambda}_h^f]; [\mathbf{v}_h, q_h, \boldsymbol{\mu}_h]) = \sum_{K \in \mathcal{T}_h^f} \Bigg[\int_K \mathbf{u}_h^f \cdot \mathbf{v}_h \, \mathrm{d}\mathbf{x} - \int_K p_h^f \, \mathrm{div}\, \mathbf{v}_h \, \mathrm{d}\mathbf{x} \\ + \int_{\partial K} p_h^f \, (\mathbf{v}_h - \boldsymbol{\mu}_h) \cdot \mathbf{n}_K \, \mathrm{d}s + \int_{\partial K} q_h \, (\mathbf{u}_h^f - \boldsymbol{\lambda}_h^f) \cdot \mathbf{n}_K \, \mathrm{d}s \\ + \beta_f \int_{\partial K} (\mathbf{u}_h^f - \boldsymbol{\lambda}_h^f) \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h) \, \mathrm{d}s - \int_K \mathrm{div}\, \mathbf{u}_h^f \, q_h \, \mathrm{d}\mathbf{x} \\ + \int_K k_f^2 \, p_h^f \, q_h \, \mathrm{d}\mathbf{x} + \frac{\delta_1}{k_f^2} \int_K (\mathrm{div}\, \mathbf{u}_h^f - k_f^2 \, p_h^f)(\mathrm{div}\, \mathbf{v}_h - k_f^2 \, q_h) \, \mathrm{d}\mathbf{x} \\ + \delta_2 \int_K (\mathbf{u}_h^f + \nabla p_h^f) \cdot (\mathbf{v}_h + \nabla q_h) \, \mathrm{d}\mathbf{x} + \frac{\delta_3}{k_f^2} \int_K \mathrm{rot}\, \mathbf{u}_h^f \, \mathrm{rot}\, \mathbf{v}_h \, \mathrm{d}\mathbf{x} \\ - \frac{i}{k_f} \int_{\partial K \cap \Gamma_f} (\boldsymbol{\lambda}_h^f \cdot \mathbf{n}_K) \, (\boldsymbol{\mu}_h \cdot \mathbf{n}_K) \, \mathrm{d}s \Bigg] \end{aligned} \tag{23}$$

and

$$\begin{aligned} F_{SHHel}([\mathbf{v}_h, q_h, \boldsymbol{\mu}_h]) = \sum_{K \in \mathcal{T}_h^f} \Bigg[\frac{\delta_1}{k_f^2} \int_K f \, (\mathrm{div}\, \mathbf{v}_h - k_f^2 \, q_h) \, \mathrm{d}\mathbf{x} - \int_K f \, q_h \, \mathrm{d}\mathbf{x} \\ + \frac{i}{k_f} \int_{\partial K \cap \Gamma_f} r \, (\boldsymbol{\mu}_h \cdot \mathbf{n}_K) \, \mathrm{d}s \Bigg], \end{aligned} \tag{24}$$

where $\mathrm{rot} = \nabla \times$ denotes the rotational operator and δ_n, with $n \in \{1, 2, 3\}$, are dimensionless stabilization parameters.

3.2 Stabilized Hybrid Formulation for the Elastic Wave Problem

To derive the hybrid formulation for the elastic wave problem in the structural domain Ω_s, we consider Eq. (7) multiplied by a weighting function and integrated by parts on each element

$K \in \mathcal{T}_h^s$, getting the following local weak formulation for $\mathbf{u}_h^s \in \mathbf{V}_h^l(\Omega_s)$, for all $\mathbf{v}_h \in \mathbf{V}_h^l(\Omega_s)$

$$\int_K \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h^s) \cdot \boldsymbol{\varepsilon}(\mathbf{v}_h)\ \mathrm{d}\mathbf{x} - \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h^s)\mathbf{n}_K \cdot \mathbf{v}_h\ \mathrm{d}s - \int_K \rho_s \omega_s^2 \mathbf{u}_h^s \cdot \mathbf{v}_h\ \mathrm{d}\mathbf{x} = \int_K \mathbf{f} \cdot \mathbf{v}_h\ \mathrm{d}\mathbf{x}.$$

The introduction of the Lagrange multiplier $\boldsymbol{\lambda}^s$ defined as the trace of $\mathbf{u}^s$, $\boldsymbol{\lambda}^s = \mathbf{u}^s|_e$, on each edge $e \in \mathcal{E}_h^s$, is made similarly to the previous problem. After the addition of a symmetrization term and a stabilization term for the multiplier on $\partial K \in \mathcal{T}_h^s$, we get the following problem:

Find $\mathbf{u}_h^s \in \mathbf{V}_h^l(\Omega_s)$ and the Lagrange multiplier $\boldsymbol{\lambda}_h^s \in \mathrm{M}_h^n(\mathcal{E}_h^s)$ such that, for all $\mathbf{v}_h \in \mathbf{V}_h^l(\Omega_s)$ and for all $\boldsymbol{\mu}_h \in \mathrm{M}_h^n(\mathcal{E}_h^s)$

$$\begin{aligned}
&\sum_{K \in \mathcal{T}_h^s} \int_K \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h^s) \cdot \boldsymbol{\varepsilon}(\mathbf{v}_h)\,\mathrm{d}\mathbf{x} - \sum_{K \in \mathcal{T}_h^s} \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h^s)\mathbf{n}_K \cdot \mathbf{v}_h\,\mathrm{d}s \\
&- \sum_{K \in \mathcal{T}_h^s} \int_K \rho_s \omega_s^2 \mathbf{u}_h^s \cdot \mathbf{v}_h\,\mathrm{d}\mathbf{x} - \sum_{K \in \mathcal{T}_h^s} \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}_K \cdot (\mathbf{u}_h^s - \boldsymbol{\lambda}_h^s)\,\mathrm{d}s \\
&+\beta_s \sum_{K \in \mathcal{T}_h^s} \int_{\partial K} (\mathbf{u}_h^s - \boldsymbol{\lambda}_h^s) \cdot \mathbf{v}_h\,\mathrm{d}s = \sum_{K \in \mathcal{T}_h^s} \int_K \mathbf{f} \cdot \mathbf{v}_h\,\mathrm{d}\mathbf{x}, \\
&\sum_{K \in \mathcal{T}_h^s} \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h^s)\mathbf{n}_K \cdot \boldsymbol{\mu}_h\,\mathrm{d}s - \beta_s \sum_{K \in \mathcal{T}_h^s} \int_{\partial K} (\mathbf{u}_h^s - \boldsymbol{\lambda}_h^s) \cdot \boldsymbol{\mu}_h\,\mathrm{d}s = 0,
\end{aligned} \tag{25}$$

where β_s is the stabilization parameter, given by

$$\beta_s = \frac{\beta_0}{h}, \quad \beta_0 > 0. \tag{26}$$

Note that the first term of Eq. (25) imposes the continuity of the normal component of the stress tensor between the elements and the second term stabilizes the Lagrange multiplier. Thus, the SHEW method supplemented by Robin boundary conditions Eq. (8) can be presented as

Find $\mathbf{u}_h^s \in \mathbf{V}_h^l(\Omega_s)$ and the Lagrange multiplier $\boldsymbol{\lambda}_h^s \in \mathrm{M}_h^n(\mathcal{E}_h^s)$ such that, for all $\mathbf{v}_h \in \mathbf{V}_h^l(\Omega_s)$ and for all $\boldsymbol{\mu}_h \in \mathrm{M}_h^n(\mathcal{E}_h^s)$

$$A_{SHEW}([\mathbf{u}_h^s, \boldsymbol{\lambda}_h^s]; [\mathbf{v}_h, \boldsymbol{\mu}_h]) = F_{SHEW}([\mathbf{v}_h, \boldsymbol{\mu}_h]), \tag{27}$$

with

$$\begin{aligned}
A_{SHEW}([\mathbf{u}_h^s, \boldsymbol{\lambda}_h^s]; [\mathbf{v}_h, \boldsymbol{\mu}_h]) = \sum_{K \in \mathcal{T}_h^s} \Bigg[&\int_K \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h^s) \cdot \boldsymbol{\varepsilon}(\mathbf{v}_h)\,\mathrm{d}\mathbf{x} - \int_K \rho_s \omega_s^2 \mathbf{u}_h^s \cdot \mathbf{v}_h\,\mathrm{d}\mathbf{x} \\
&- \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h^s)\mathbf{n}_K \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h)\,\mathrm{d}s - \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}_K \cdot (\mathbf{u}_h^s - \boldsymbol{\lambda}_h^s)\,\mathrm{d}s \\
&+ \beta_s \int_{\partial K} (\mathbf{u}_h^s - \boldsymbol{\lambda}_h^s) \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h)\,\mathrm{d}s + \int_{\partial K \cap \Gamma_s} i\,\mathbf{A}\,\boldsymbol{\lambda}_h^s \cdot \boldsymbol{\mu}_h\,\mathrm{d}s \Bigg]
\end{aligned} \tag{28}$$

and

$$F_{SHEW}([\mathbf{v}_h, \boldsymbol{\mu}_h]) = \sum_{K \in \mathcal{T}_h^s} \left[\int_K \mathbf{f} \cdot \mathbf{v}_h\,\mathrm{d}\mathbf{x} + \int_{\partial K \cap \Gamma_s} \mathbf{g} \cdot \boldsymbol{\mu}_h\,\mathrm{d}s \right]. \tag{29}$$

3.3 Stabilized Hybrid Formulation for Acoustic Fluid-Structure Interaction

In order to generate a coupled hybrid method for the fluid structure interaction we use the stabilized hybrid formulations Eq. (22) for the fluid domain and Eq. (27) for the structure domain. Moreover, the interface fluid/structure conditions (Eqs. (14)-(15)) are naturally imposed by the Lagrange multiplier. Thus, on the edges $e \in \mathcal{E}_h^{fs}$ that compose the interface Γ_{fs}, we have for the fluid domain

$$\sum_{K\in\mathcal{T}_h^f} \left[\int_{\Gamma_{sf}} p_h^f \,(\mathbf{v}_h - \boldsymbol{\mu}_h)\cdot \mathbf{n}_f \,\mathrm{d}s + \int_{\Gamma_{sf}} q_h \,(\mathbf{u}_h^f - \boldsymbol{\lambda}_h^f)\cdot \mathbf{n}_f \,\mathrm{d}s \right. \\ \left. + \beta_f \int_{\Gamma_{sf}} (\mathbf{u}_h^f - \boldsymbol{\lambda}_h^f)\cdot(\mathbf{v}_h - \boldsymbol{\mu}_h)\,\mathrm{d}s \right], \tag{30}$$

and for solid domain

$$\sum_{K\in\mathcal{T}_h^s} \left[-\int_{\Gamma_{sf}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h^s)\mathbf{n}_s \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h)\,\mathrm{d}s - \int_{\Gamma_{sf}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}_s \cdot (\mathbf{u}_h^s - \boldsymbol{\lambda}_h^s)\,\mathrm{d}s \right. \\ \left. + \beta_s \int_{\Gamma_{sf}} (\mathbf{u}_h^s - \boldsymbol{\lambda}_h^s)\cdot(\mathbf{v}_h - \boldsymbol{\mu}_h)\,\mathrm{d}s \right]. \tag{31}$$

Choosing the Lagrange multiplier $\boldsymbol{\lambda}_h^f$, associated to the Helmholtz velocity and stabilized by the parameter β_f, on the interface Γ_{fs}, the terms in (31) must adapt to satisfy the interface conditions from Eqs. (14)-(15). Thus considering only the normal component of the term multiplied by β_s, including the chosen multiplier on the interface and using identity

$$\boldsymbol{\lambda}_h^s \cdot \mathbf{n}_s = \frac{1}{\rho_f \omega_s^2} \boldsymbol{\lambda}_h^f \cdot \mathbf{n}_s \qquad \text{on} \quad \Gamma_{fs}, \tag{32}$$

obtained from the equation (14) in Eq. (31), we derive an interface condition able to naturally couple the two media

$$\sum_{K\in\mathcal{T}_h^s} \left[-\int_{\Gamma_{sf}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h^s)\mathbf{n}_s \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h)\,ds \right. \\ \left. + \beta_f \int_{\Gamma_{sf}} \left(\mathbf{u}_h^s - \frac{1}{\rho_f \omega_s^2} \boldsymbol{\lambda}_h^f \right) \cdot \mathbf{n}_s \,(\mathbf{v}_h - \boldsymbol{\mu}_h)\cdot \mathbf{n}_s \,ds \right]. \tag{33}$$

Using the interface condition Eq. (33) to connect the SHHel method with SHEW method we introduce the Stabilized Hybrid method for Acoustic Fluid-Structure interaction (*SHFS*), as follows:

Find $[\mathbf{u}_h^i, p_h^f] \in \mathbf{V}_h^l(\Omega_i) \times Q_h^m(\Omega_f)$, *with* $i = f, s$, *and the Lagrange multipliers* $\boldsymbol{\lambda}_h^i \in \mathbf{M}_h^n(\mathcal{E}_h^i)$ *such that, for all* $[\mathbf{v}_h, q_h] \in \mathbf{V}_h^l(\Omega) \times Q_h^m(\Omega_f)$ *and* $\boldsymbol{\mu}_h \in \mathbf{M}_h^n(\mathcal{E}_h)$

$$A_{SHFS}([\mathbf{u}_h^f, \mathbf{u}_h^s, p_h^f, \boldsymbol{\lambda}_h^f, \boldsymbol{\lambda}_h^s]; [\mathbf{v}_h, q_h, \boldsymbol{\mu}_h]) = F_{SHFS}([\mathbf{v}_h, q_h, \boldsymbol{\mu}_h]), \tag{34}$$

with

$$A_{SHFS}([\mathbf{u}_h^f, \mathbf{u}_h^s, p_h^f, \boldsymbol{\lambda}_h^f, \boldsymbol{\lambda}_h^s]; [\mathbf{v}_h, q_h, \boldsymbol{\mu}_h]) = A_{SHHel}([\mathbf{u}_h^f, p_h^f, \boldsymbol{\lambda}_h^f]; [\mathbf{v}_h^f, q_h, \boldsymbol{\mu}_h^f])$$

$$+ A_{SHEW}([\mathbf{u}_h^s, \boldsymbol{\lambda}_h^s]; [\mathbf{v}_h^s, \boldsymbol{\mu}_h^s])$$

$$+ \sum_{K \in \mathcal{T}_h^s} \left[- \int_{\Gamma_{sf}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h^s)\mathbf{n}_s \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h)\, \mathrm{d}s \right.$$

$$\left. + \beta_f \int_{\Gamma_{sf}} \left(\mathbf{u}_h^s - \frac{1}{\rho_f \omega_s^2} \boldsymbol{\lambda}_h^f \right) \cdot \mathbf{n}_s \, (\mathbf{v}_h - \boldsymbol{\mu}_h) \cdot \mathbf{n}_s \, \mathrm{d}s \right] \tag{35}$$

and

$$F_{SHFS}([\mathbf{v}_h, q_h, \boldsymbol{\mu}_h]) = F_{SHHel}([\mathbf{v}_h, q_h, \boldsymbol{\mu}_h]) + F_{SHEW}([\mathbf{v}_h, \boldsymbol{\mu}_h]). \tag{36}$$

4 SOLVING METHODOLOGY

We start with the solving methodology for the SHHel method, given by Eq. (22). In order to solve the proposed formulation, we eliminate the degrees-of-freedom for the variables $\mathbf{u}_h^f$ and p_h^f at the element level in favor of the degrees-of-freedom for the multiplier $\boldsymbol{\lambda}_h^f$, leading to a global system in the multipliers only. Approximated solutions for the variables $\mathbf{u}_h^f$ and p_h^f can then be sought through a set of local problems, each one defined on a element $K \in \mathcal{T}_h^f$. The problem can be written as

Find $[\mathbf{u}_h^f, p_h^f] \in \mathbf{V}_h^l(\Omega_f) \times Q_h^m(\Omega_f)$ *and the Lagrange multiplier* $\boldsymbol{\lambda}_h^f \in \mathrm{M}_h^n(\mathcal{E}_h^f)$ *such that, for all* $[\mathbf{v}_h, q_h] \in \mathbf{V}_h^l(\Omega_f) \times Q_h^m(\Omega_f)$ *and* $\boldsymbol{\mu}_h \in \mathrm{M}_h^n(\mathcal{E}_h^f)$

$$a_K([\mathbf{u}_h^f, p_h^f], [\mathbf{v}_h, q_h]) + b_K(\boldsymbol{\lambda}_h^f, [\mathbf{v}_h, q_h]) = f_K([\mathbf{v}_h, q_h]), \quad \forall K \in \mathcal{T}_h^f, \tag{37}$$

$$\sum_{K \in \mathcal{T}_h} b_K^T([\mathbf{u}_h^f, p_h^f], \boldsymbol{\mu}) + \sum_{K \in \mathcal{T}_h} c_K(\boldsymbol{\lambda}_h^f, \boldsymbol{\mu}) = g_K(\boldsymbol{\mu}_h), \tag{38}$$

with

$$
\begin{aligned}
a_K([\mathbf{u}_h^f, p_h^f], [\mathbf{v}_h, q_h]) = & \int_K \mathbf{u}_h^f \cdot \mathbf{v}_h \, \mathrm{d}\mathbf{x} - \int_K p_h^f \, \operatorname{div} \mathbf{v}_h \, \mathrm{d}\mathbf{x} + \int_K k_f^2 \, p_h^f \, q_h \, \mathrm{d}\mathbf{x} \\
& + \int_{\partial K} p_h^f \, (\mathbf{v}_h \cdot \mathbf{n}_K) \, \mathrm{d}s + \int_{\partial K} q_h \, (\mathbf{u}_h^f \cdot \mathbf{n}_K) \, \mathrm{d}s \\
& - \int_K \operatorname{div} \mathbf{u}_h^f \, q_h \, \mathrm{d}\mathbf{x} + \beta_f \int_{\partial K} \mathbf{u}_h^f \cdot \mathbf{v}_h \, \mathrm{d}s \\
& + \frac{\delta_1}{k_f^2} \int_K (\operatorname{div} \mathbf{u}_h^f - k_f^2 \, p_h^f)(\operatorname{div} \mathbf{v}_h - k_f^2 \, q_h) \, \mathrm{d}\mathbf{x} \\
& + \delta_2 \int_K (\mathbf{u}_h^f + \nabla p_h^f) \cdot (\mathbf{v}_h + \nabla q_h) \, \mathrm{d}\mathbf{x} \\
& + \frac{\delta_3}{k_f^2} \int_K \operatorname{rot} \mathbf{u}_h^f \operatorname{rot} \mathbf{v}_h \, \mathrm{d}\mathbf{x},
\end{aligned}
$$

$$
b_K(\boldsymbol{\lambda}_h^f, [\mathbf{v}_h, q_h]) = - \int_{\partial K} q_h \, (\boldsymbol{\lambda}_h^f \cdot \mathbf{n}_K) \, \mathrm{d}s - \beta_f \int_{\partial K} \boldsymbol{\lambda}_h^f \cdot \mathbf{v}_h \, \mathrm{d}s,
$$

$$
c_K(\boldsymbol{\lambda}_h^f, \boldsymbol{\mu}_h) = \beta_f \int_{\partial K} \boldsymbol{\lambda}_h^f \cdot \boldsymbol{\mu}_h \, \mathrm{d}s - \frac{i}{k_f} \int_{\partial K \cap \Gamma_f} (\boldsymbol{\lambda}_h^f \cdot \mathbf{n}_K) \, (\boldsymbol{\mu}_h \cdot \mathbf{n}_K) \, \mathrm{d}s,
$$

$$
f_K([\mathbf{v}_h, q_h]) = \frac{\delta_1}{k_f^2} \int_K f \, (\operatorname{div} \mathbf{v}_h - k_f^2 \, q_h) \, \mathrm{d}\mathbf{x} - \int_K f \, q_h \, \mathrm{d}\mathbf{x},
$$

$$
g_K(\boldsymbol{\mu}_h) = \frac{i}{k_f} \int_{\partial K \cap \Gamma_f} r \, (\boldsymbol{\mu}_h \cdot \mathbf{n}_K) \, \mathrm{d}s.
$$

The solving strategy for both the SHEW and SHFS methods is analogous.

Rewriting Eqs. (37) and (38) in a matrix form, we obtain

$$
\mathbf{A}_K \mathbf{U} + \mathbf{B}_K \boldsymbol{\Lambda} = \mathbf{F}_K, \quad \forall K \in \mathcal{T}_h \tag{39}
$$

$$
\sum_{K \in \mathcal{T}_h} \mathbf{B}_K^T \mathbf{U} + \sum_{K \in \mathcal{T}_h} \mathbf{C}_K \boldsymbol{\Lambda} = \mathbf{0}, \tag{40}
$$

where

$$
\text{for SHHel:} \;\; \mathbf{U} = \begin{pmatrix} u_x^f \\ u_y^f \\ p^f \end{pmatrix} \quad \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_x^f \\ \lambda_y^f \end{pmatrix}; \tag{41}
$$

$$
\text{for SHEW:} \;\; \mathbf{U} = \begin{pmatrix} u_x^s \\ u_y^s \end{pmatrix} \quad \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_x^s \\ \lambda_y^s \end{pmatrix}. \tag{42}
$$

For the hybrid coupled SHFS method we have the variables Eq. (41) in the fluid domain Ω_f and Eq. (42) in the structure domain Ω_s. On the interface Γ_{fs} we adopt $\mathbf{\Lambda} = (\lambda_x^f, \lambda_y^f)^T$.

Given that $\mathbf{A}_K$ is positive definite, we solve Eq. (39) to obtain

$$\mathbf{U} = \mathbf{A}_K^{-1}(\mathbf{F}_K - \mathbf{B}_K\mathbf{\Lambda}), \quad \forall K \in \mathcal{T}_h. \tag{43}$$

Replacing Eq. (43) in Eq. (40), we obtain the global system in $\mathbf{\Lambda}$ only, as follows

$$\sum_{K\in\mathcal{T}_h}(\mathbf{C}_K - \mathbf{B}_K^T\mathbf{A}_K^{-1}\mathbf{B}_K)\mathbf{\Lambda} = -\sum_{K\in\mathcal{T}_h}\mathbf{B}_K^T\mathbf{A}_K^{-1}\mathbf{F}_K. \tag{44}$$

After solving the global Eq. (44), the vector $\mathbf{U}$ can be obtained from Eq. (43).

5 NUMERICAL RESULTS

In this section we present numerical results for two of the proposed formulations: SHHel and SHEW.

5.1 Numerical Results in Fluid Domain - SHHel method

In the numerical experiments we consider a domain $\Omega = [0,1]\times[0,1]$, $k_f = 12$, $\theta = \pi/6$ and $f(x,y) = 0$ to develop the following exact solution

$$p_f(x,y) = \cos[k_f(x\cos\theta + y\sin\theta)] + i\sin[k_f(x\cos\theta + y\sin\theta)]. \tag{45}$$

A comparative study of the convergence rates obtained with SHHel, the Local Projection (LP) [1] and the Interpolant (I) for variables $\mathbf{u}_h^f$, p_h^f and $\boldsymbol{\lambda}_h^f$ is presented in Figs. 2–4. The approximate solutions have been obtained using uniform meshes of $(10{\cdot}j)\times(10{\cdot}j)$, with $j = 2,3,4,5,6,7,8$, elements. The plots present approximations in L^2-norm. Furthermore, in all simulations we fixed

$$\delta_1 = 0,5; \quad \delta_2 = -0,5; \quad \delta_3 = 0,5.$$

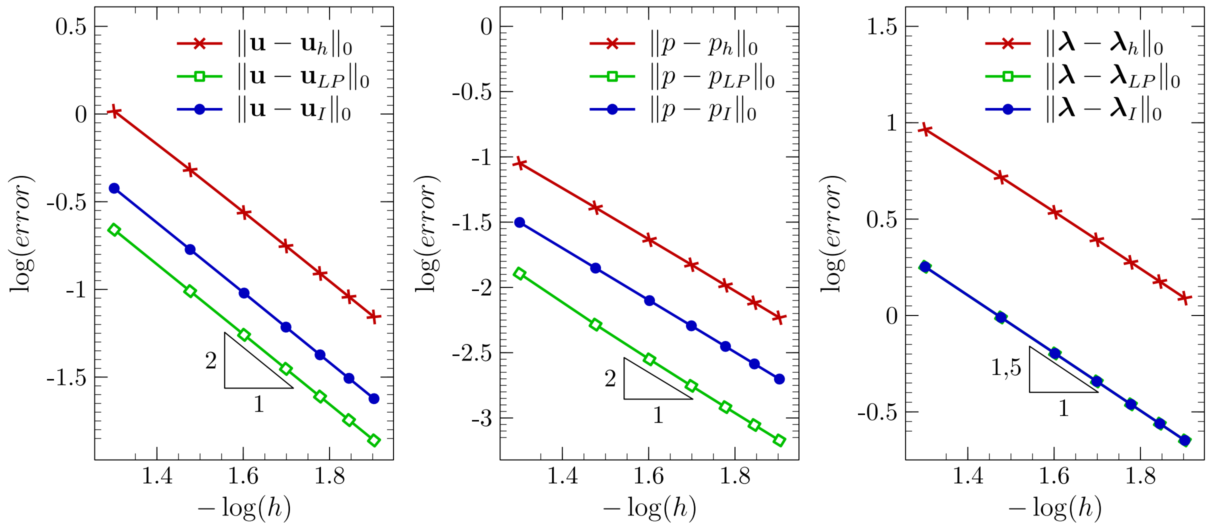
For Fig. 2 we present a h-convergence using the SHHel approximations on quadrilateral elements $\mathbb{Q}_1\mathbb{Q}_1 - p_1$. In Fig. 3 we present a h-convergence using the SHHel approximations on quadrilaterals elements $\mathbb{Q}_2\mathbb{Q}_2 - p_2$. In Fig. 4 show results of convergence study using a fixed 20×20 uniform mesh and varying the degree of the polynomial approximations by setting $l = m = n = 1,2,3,4,5$ sequentially.

5.2 Numerical Results in Solid Domain - SHEW method

For the Elastic Wave problem, numerical experiments are developed in a domain $\Omega = [0,1]\times[0,1]$, where the values for the density constant, Poisson's ratio, Young's modulus and frequency, are given by: $\rho_s = 1$, $E = 1$, $\nu = 0.3$ and $\omega_s = 20$. Moreover we adopt $k_p = 17.23$, $k_s = 32.25$ and $\theta = \pi/6$ to derive the following analytical solution

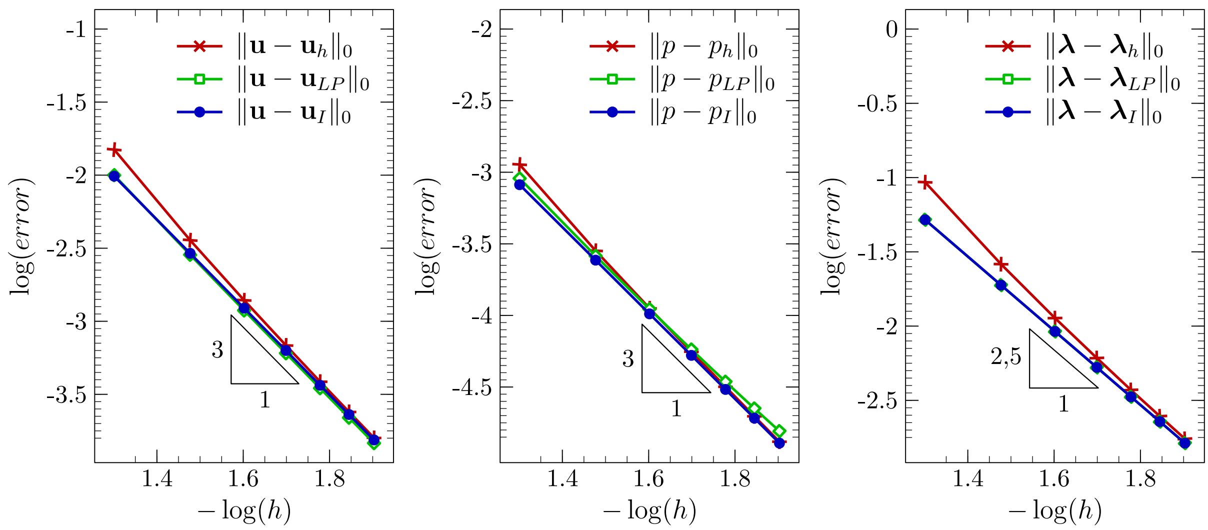
$$\begin{aligned}\mathbf{u}_s(x,y) &= (\cos\theta,\sin\theta)^T\exp[ik_p(x\cos\theta + y\sin\theta)]\\ &+ (-\sin\theta,\cos\theta)^T\exp[ik_s(x\cos\theta + y\sin\theta)].\end{aligned} \tag{46}$$

[1] The Local Projection is obtained using the exact solution (45) for the multiplier in the system (43).



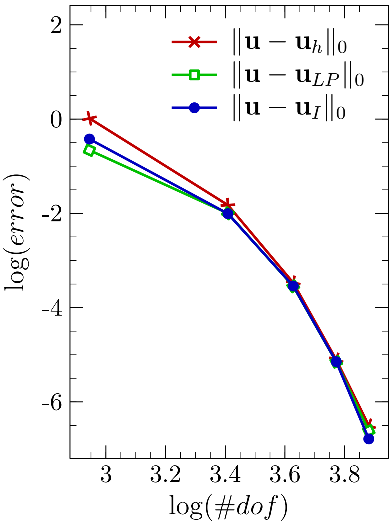
(a) $\mathbf{u}_h^f$ (b) p_h^f (c) $\boldsymbol{\lambda}_h^f$

Figure 2: Helmholtz: h-Convergence for the $\mathbf{u}_h^f$, p_h^f and $\boldsymbol{\lambda}_h^f$ approximations by the SHHel hybrid method (h), Local Projection (LP) and Interpolant (I). Error in the L^2-norm for quadrilaterals elements $\mathbb{Q}_1\mathbb{Q}_1-p_1$.

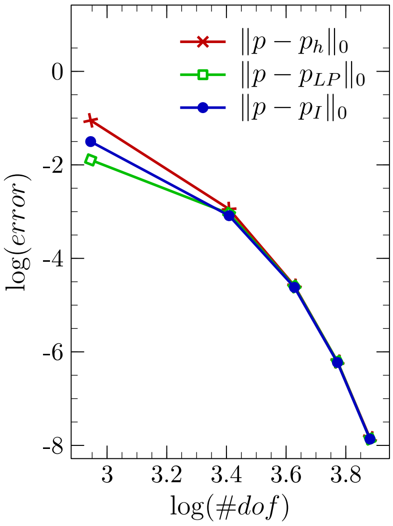


(a) $\mathbf{u}_h^f$ (b) p_h^f (c) $\boldsymbol{\lambda}_h^f$

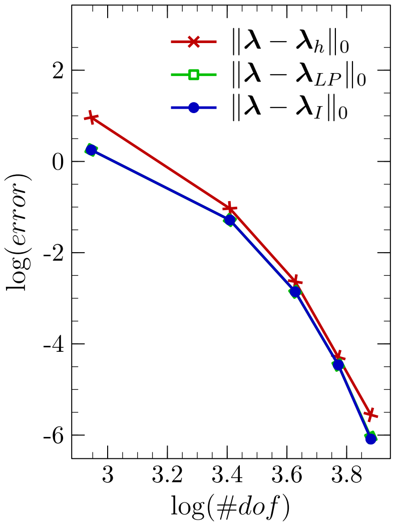
Figure 3: Helmholtz: h-Convergence for the $\mathbf{u}_h^f$, p_h^f and $\boldsymbol{\lambda}_h^f$ approximations by the SHHel hybrid method (h), Local Projection (LP) and Interpolant (I). Error in the L^2-norm for quadrilaterals elements $\mathbb{Q}_2\mathbb{Q}_2-p_2$.



(a) $\mathbf{u}_h^f$



(b) p_h^f



(c) $\boldsymbol{\lambda}_h^f$

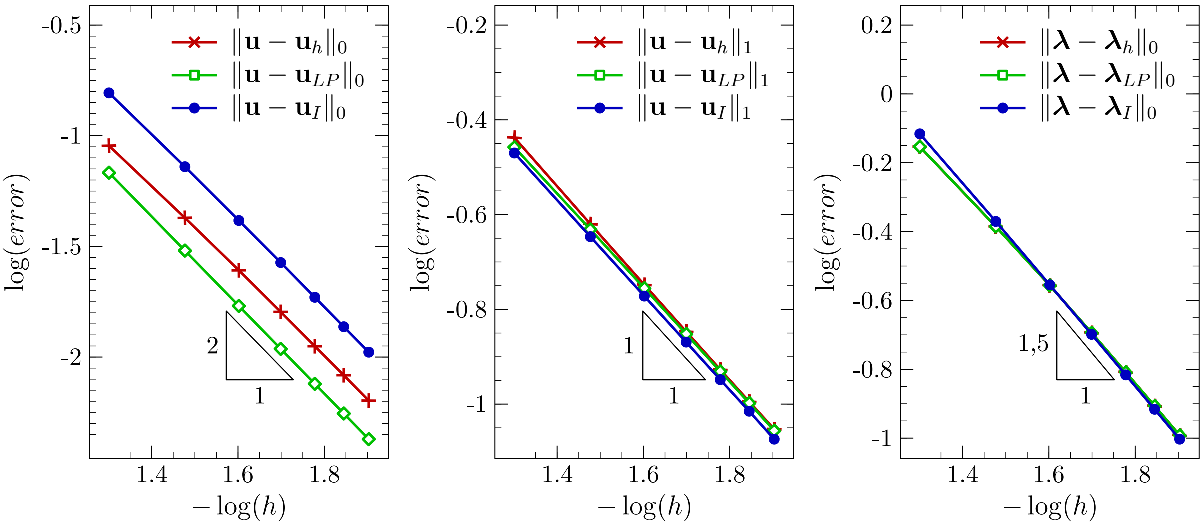
Figure 4: Helmholtz: p-Convergence for the $\mathbf{u}_h^f$, p_h^f and $\boldsymbol{\lambda}_h^f$ approximations by the SHHel hybrid method (h), Local Projection (LP) and Interpolant (I). Error in the L^2-norm for fixed 20×20 elements mesh.

In tests we compare the convergence rates obtained with SHEW, the Local Projection (LP)[2] and the Interpolant (I) for variables $\mathbf{u}_h^s$, $\nabla \mathbf{u}_h^s$ and $\boldsymbol{\lambda}_h^s$, Figs. 5–7, employing uniform meshes of $(10\cdot j)\times(10\cdot j)$, with $j = 2, 3, 4, 5, 6, 7, 8$, elements. For stability parameter Eq. (26) we choose $\beta_0 = 5$ for $\mathbb{Q}_1 - p_1$ approximations and $\beta_0 = 12$ for $\mathbb{Q}_2 - p_2$ approximations. In Fig. 5 we present a h-convergence using the SHEW approximations on quadrilateral elements $\mathbb{Q}_1 - p_1$. In Fig. 6 we present a h-convergence using the SHEW approximations on quadrilateral elements $\mathbb{Q}_2 - p_2$. In Fig. 7 much more accurate solutions are obtained by increasing the degree of the polynomial approximations, where p-convergence results are presented using a fixed 20×20 uniform mesh and varying the degree of the polynomial approximations by setting $l = m = n = 1, 2, 3, 4, 5$ sequentially, in this case we adopt the respective values of $\beta_0 = 5, 12, 20, 45, 60$.

6 CONCLUSIONS

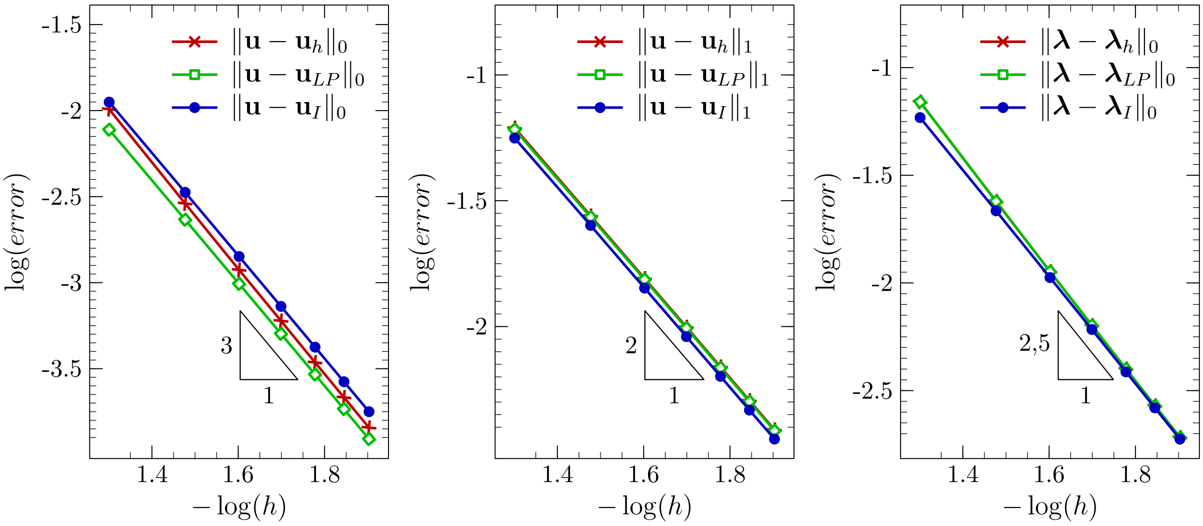
We developed a Stabilized dual Hybrid mixed finite element method for the Helmholtz problem (SHHel), a Stabilized primal Hybrid method for the time-harmonic Elastic Wave problem (SHEW) and a Stabilized Hybrid method for acoustic Fluid-Structure interaction (SHFS). The continuity of this methods are imposed via Lagrange multipliers identified as the trace of the velocity/displacement field only on the edges of the elements leading to a set of local problems defined at the element level and a global problem in the multiplier only. Then, the global problem, involving only the degrees-of-freedom of the multiplier is solved leading to the approximate solution of the multiplier, which is plugged into the local problems to recover the discontinuous approximation of the variables. The interface of the acoustic fluid-structure problem is naturally imposed by the Lagrange multipliers.

[2]The Local Projection is obtained using the exact solution (46) for the multipliers in the system (43).



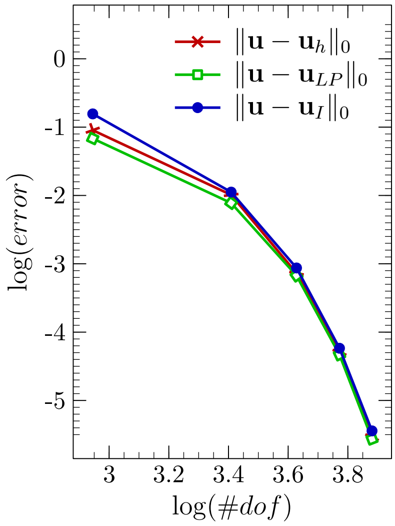
(a) $\mathbf{u}_h^s$ (b) $\nabla\mathbf{u}_h^s$ (c) $\boldsymbol{\lambda}_h^s$

Figure 5: Elastic Wave: h-Convergence for the $\mathbf{u}_h^s$, $\nabla\mathbf{u}_h^s$ and $\boldsymbol{\lambda}_h^s$ approximations by the SHEW hybrid method (h), Local Projection (LP) and Interpolant (I). Error in the L^2-norm (left and right) and H^1-seminorm for quadrilaterals elements $\mathbb{Q}_1 - p_1$.

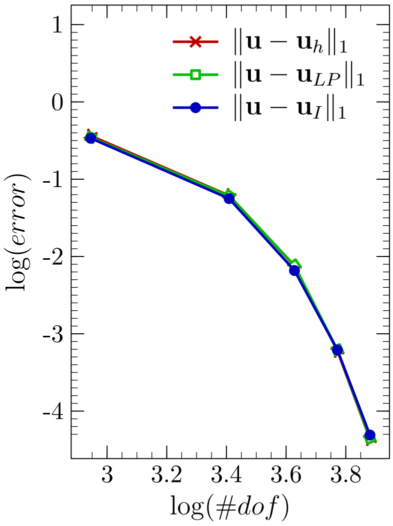


(a) $\mathbf{u}_h^s$ (b) $\nabla\mathbf{u}_h^s$ (c) $\boldsymbol{\lambda}_h^s$

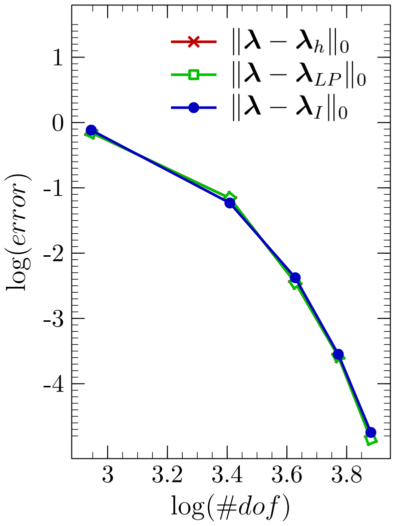
Figure 6: Elastic Wave: h-Convergence for the $\mathbf{u}_h^s$, $\nabla\mathbf{u}_h^s$ and $\boldsymbol{\lambda}_h^s$ approximations by the SHEW hybrid method (h), Local Projection (LP) and Interpolant (I). Error in the L^2-norm (left and right) and H^1-seminorm for quadrilaterals elements $\mathbb{Q}_2 - p_2$.



(a) $\mathbf{u}_h^s$



(b) $\nabla\mathbf{u}_h^s$



(c) $\boldsymbol{\lambda}_h^s$

Figure 7: Elastic Wave: p-Convergence for the $\mathbf{u}_h^s$, $\nabla\mathbf{u}_h^s$ and $\boldsymbol{\lambda}_h^s$ approximations by the SHEW hybrid method (h), Local Projection (LP) and Interpolant (I). Error in the L^2-norm(left and right) and H^1-seminorm for fixed 20×20 elements mesh.

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REFERENCES

M. R. Correa and A. F. D. Loula, 2008. Unconditionally stable mixed finite element methods for Darcy flow. *Computer Methods in Applied Mechanics and Engineering*, vol. 197, n. 17–18, pp. 1525–1540.

A. Craggs, 1972. The use of simple three-dimensional acoustic finite elements for determining the natural modes and frequencies of complex shaped enclosures. *Journal of Sound and Vibration*, vol. 23, n. 3, pp. 331–339.

G.M.L. Gladwell, 1966. A variational formulation of damped acousto structural vibration problems. *Journal of Sound and Vibration*, vol. 4, n. 2, pp. 172–186.

G.M.L. Gladwell and V. Mason, 1971. Variational finite element calculation of the acoustic response of a rectangular panel. *Journal of Sound and Vibration*, vol. 14, n. 1, pp. 115–135.

José A. González, K.C. Park, I. Lee, C.A. Felippa, and R. Ohayon, 2012. Partitioned vibration analysis of internal fluid-structure interaction problems. *International Journal for Numerical Methods in Engineering*, vol. 92, n. 3, pp. 268–300.

I. Harari and T. J. R. Hughes, 1992. Galerkin/last-squares finite element methodsfor the reducedwave equations with nonreflecting bondery conditions in unbonded domains. *Comput. Methods Appl. Mech. Engrg*, vol. 98, pp. 411–454.

I. Igreja. *Métodos de Elementos Finitos Híbridos Estabilizados para Escoamentos de Stokes, Darcy e Stokes-Darcy Acoplados*. PhD thesis, Laboratório Nacional de Computação Científica - LNCC, 2015.

I. Igreja, Abimael F. D. Loula, and C. O. Faria, 2014. A new hybridized mixed finite element method for heterogenous porous media flow. *Proceedings of the XXXV Iberian Latin-American Congress on Computational Methods in Engineering (CILAMCE)*.

I. Igreja, C. O. Faria, and A. F. D. Loula, 2015. Métodos de elementos finitos mistos híbridos para o escoamento de Stokes-Darcy acoplado. *Proceedings Congresso de Métodos Numéricos em Engenharia - CMN, Lisboa/Portugal*.

A. F. D. Loula, 2011. Stabilized hybrid and mixed finite element methods for helmholtz problems. In M. Papadrakakis, M. Fragiadakis, and V. Plevris, editors, *Proc. 3rd. ECOOMAS Thematic Conference on Computational Methods in Structural and Earthquake Engineering (Corfu, Greece, 25–28 May 2011)*, pp. 3210–3222.

C. J. Luke and P. A. Martin, 1995. Fluid–solid interaction: Acoustic scattering by a smooth elastic obstacle. *SIAM Journal on Applied Mathematics*, vol. 55, n. 4, pp. 904–922.

P. Monk and D. Wang, 1999. A least-squares method for the helmholtz equation. *Comput Method Appl Mech Eng*, vol. 175, pp. 121–136.

D.J. Nefske, J.A. Wolf Jr, and L.J. Howell, 1982. Structural-acoustic finite element analysis of the automobile passenger compartment: A review of current practice. *Journal of Sound and Vibration*, vol. 80, n. 2, pp. 247–266.

Michael R. Ross, Carlos A. Felippa, K.C. Park, and Michael A. Sprague, 2008. Treatment of acoustic fluid-structure interaction by localized lagrange multipliers: Formulation. *Computer Methods in Applied Mechanics and Engineering*, vol. 197, n. 33–40, pp. 3057–3079.

Michael R. Ross, Michael A. Sprague, Carlos A. Felippa, and K.C. Park, 2009. Treatment of acoustic fluid–structure interaction by localized lagrange multipliers and comparison to alternative interface-coupling methods. *Computer Methods in Applied Mechanics and Engineering*, vol. 198, n. 9–12, pp. 986–1005.

W. M. Vicente, R. Picelli, R. Pavanello, and Y. M. Xie, 2015. Topology optimization of frequency responses of fluid–structure interaction systems. *Finite Elements in Analysis and Design*, vol. 98, pp. 1–13.

G. H. Yoon, J. S. Jensen, and O. Sigmund, 2007. Topology optimization of acoustic–structure interaction problems using a mixed finite element formulation. *International Journal for Numerical Methods in Engineering*, vol. 70, n. 9, pp. 1049–1075.

O. C. Zienkiewicz and P. Bettess, 1978. Fluid-structure dynamic interaction and wave forces. an introduction to numerical treatment. *International Journal for Numerical Methods in Engineering*, vol. 13, n. 1, pp. 1–16.